Surface critical behavior of random systems: Ordinary transition

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We calculate the surface critical exponents of the ordinary transition occurring in semi-infinite, quenched dilute Ising-like systems. This is done by applying the field theoretic approach directly in $d=3$ dimensions up to the two-loop approximation as well as in $4-\varepsilon$ dimensions. At $d=4-\varepsilon$ we extend, up to the next-to-leading order, the previous first-order results of the $\sqrt{\varepsilon}$ expansion by Ohno and Okabe [Phys. Rev. B **46**, 5917 (1992)]. In both cases numerical estimates for surface exponents are computed using Padé approximants extrapolating the perturbation theory expansions. The obtained results indicate that the critical behavior of semi-infinite systems with quenched bulk disorder is characterized by the new set of surface critical exponents.

I. INTRODUCTION

The critical behavior of quenched random systems undergoing continuous phase transitions is of great interest. It is well known that the critical behavior of ideal pure bulk substances may be changed by introducing disorder. A prominent result in the theory of quenched disordered systems is the Harris criterion [1], which states that the presence of disorder is relevant for those pure systems which have a positive specific heat exponent α. Thus, in the class of $O(N)$ symmetric N-vector models in d space dimensions Ising model is the one of primary interest, having $\alpha(d) \geq 0$. The marginal value, $\alpha = 0$, corresponds to $d = 2$. In this case the small amount of disorder produces a marginal perturbation. This results in logarithmic corrections to the power-law singularities without modification of the critical exponents with respect to the pure system (for reviews, see Refs. [2,3]).

In three-dimensional disordered Ising systems a new set of modified critical exponents appears. This was confirmed by renormalization group (RG) calculations [4–7], experiments [8–10], and Monte-Carlo simulations [11,12]. After pioneering investigations using the Wilson's RG and ε expansion [4–7,13,14], scaling-field method [15], and the massive field theory in three dimensions [16–19], there has been a great renewed interest to the subject in the last few years [20–24]. An interesting relation between random Ising model and ''N-colored'' tethered membranes has been given in Ref. [25]. Various aspects of physics of quenched disordered systems are the subject of recent extensive MC investigations [26–28].

The presence of boundaries, which are inevitable in real systems, leads to the new surface physics. General reviews on surface critical phenomena are given in Refs. [29–31]. It is now well known that there are several surface universality classes defining the critical behavior in the vicinity of boundaries, at temperatures close to the bulk critical point. Each surface universality class is defined both by the bulk universality class and specific properties of a given boundary.

What happens with the surface critical behavior, when the quenched disorder is introduced? First, the defects may be localized only at the boundary. In Ref. [32], the relevance-irrelevance criteria of the Harris type concerning the weak surface disorder have been worked out. In the case of the ordinary transition (which corresponds to the free boundary conditions), it has been demonstrated that for $d > 2$ the weak surface randomness is an irrelevant perturbation. The same conclusion follows from rigorous arguments [33] and Monte Carlo simulations [34]. There is strong evidence that a small amount of quenched surface disorder is irrelevant at the special transition as well [32,33,35].

Second, the random quenched disorder may be introduced in the bulk, say, of a semi-infinite system bounded by a plane surface. This will, in general, shift the critical temperature of the bulk phase transition, and drive the system to another, ''random'' fixed point. According to the Harris criterion, Ising systems are of main concern here. The change of the universality class of the bulk will affect the critical behavior of the bounding surface. Thus one will expect the new surface critical exponents to appear. And, in view of the aforementioned properties of boundaries, these new surface exponents should emerge irrespective of the presence of the disorder just at the boundary itself.

This is a situation typical for $d > 2$. The problem of a RG calculation of ''disorder-induced'' surface critical exponents was first addressed by Ohno and Okabe in 1992 [36] by using the ε expansion about the upper critical dimension 4. In the marginal case $d = 2$, the boundary critical behavior of random systems has been studied very recently in a series of papers [37–39].

In the present paper, we present an attempt to go beyond the approaches and approximations employed in [36] in calculating the surface critical exponents of random semi-infinite Ising-like models. Our main calculations are performed directly at $d = 3$ using the massive field theory

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approach [40–43,35] in the two-loop approximation. Moreover, we extend, up to the next-to-leading order of $\sqrt{\varepsilon}$ expansion, the previous first-order results of Ohno and Okabe [36]. In both cases, numerical estimates for surface critical exponents of the ordinary transition are evaluated using Padé approximants improving the direct perturbation theory expansions. The obtained values of the surface critical exponents are consistent with the results obtained by Ohno and Okabe [36] and confirm that the semi-infinite systems with quenched bulk disorder are characterized by a new set of the surface critical exponents.

II. MODEL

The description of the surface critical behavior at the ordinary transition can be formulated in terms of the effective Landau-Ginzburg-Wilson Hamiltonian

$$H[\varphi] = \int_0^\infty dz \int d^{d-1}r \left[\frac{1}{2} |\nabla \varphi|^2 + \frac{1}{2} \tau_0 \varphi^2 + \frac{1}{4!} v_0 \varphi^4 \right]. \quad (2.1)$$

The d-dimensional spatial integration is extended over a half-space $\mathbb{R}_+^d \equiv \{x = (r,z) \in \mathbb{R}^d | r \in \mathbb{R}^{d-1}, z \geq 0\}$ bounded by a plane surface at $z = 0$. Here $\varphi = \varphi(x)$ is a continuous scalar field corresponding to the one-component order parameter of an original Ising system. The surface is considered to be free, and hence the field $\varphi(x)$ satisfies Dirichlet boundary conditions at $z = 0$, $\varphi(r,0) = 0$ [44,30].

One of the possibilities to introduce disorder into the model is to assume that the parameter τ_0 incorporates local random temperature fluctuations $\delta\tau(x)$ via $\tau_0 = m_0^2 + \delta\tau(x)$. Here m_0^2 is a "bare mass" representing linear temperature deviations from the mean-field critical temperature, and the random variable $\delta\tau(x)$ has the properties $\langle \delta\tau(x) \rangle_{\text{conf}} = 0$ and $\langle \delta\tau(x)\delta\tau(x') \rangle_{\text{conf}} = \Delta \delta(x - x')$ with $\Delta > 0$. Angular brackets with the subscript "conf" denote configurational averaging over quenched disorder which should be implemented on the level of the free energy [45,46].

Ohno and Okabe [36] analyzed the above random model by direct averaging over disorder using the method originally introduced by Lubensky [6]. Alternatively, the configurational averaging of the free energy can be performed using the replica trick

$$F = -T \lim_{n \to 0} \frac{1}{n} (\langle Z^n \rangle_{\text{conf}} - 1), \quad (2.2)$$

where Z is the partition of a configuration given by the Boltzman weight $e^{-H[\varphi]}$, as it was first done in the RG calculations by Grinstein and Luther [7]. We shall use this last possibility to treat randomness.

Employing the replica trick leads to the effective LGW Hamiltonian with cubic anisotropy defined in the semi-infinite space

$$H[\vec{\varphi}] = \int_0^\infty dz \int d^{d-1}r \left[\frac{1}{2} |\nabla \vec{\varphi}|^2 + \frac{1}{2} m_0^2 |\vec{\varphi}|^2 \right.$$
$$\left. + \frac{1}{4!} v_0 \sum_{i=1}^n \varphi_i^4 + \frac{1}{4!} u_0 (|\vec{\varphi}|^2)^2 \right], \quad (2.3)$$

where $\vec{\varphi}(x)$ is an n-vector field with the components $\varphi_i(x)$, $i = 1, \ldots, n$. This last effective Hamiltonian is appropriate for the description of the ordinary transition of random systems in the replica limit $n \to 0$. The $\mathcal{O}(n)$ symmetric term arises due to the random averaging in Eq. (2.2) via cumulant expansion and thus its coupling constant $u_0 \propto -\Delta < 0$.

III. CORRELATION FUNCTIONS AND THEIR RENORMALIZATIONS

The model defined in Eq. (2.3) is translationally invariant in directions parallel to the boundary. Thus it is often useful to perform Fourier transformations in $(d-1)$-dimensional subspace with respect to "parallel" coordinates r. We shall denote the associated parallel momenta as p. In the perpendicular direction the coordinate z is retained. Sometimes, in the perturbative calculations it is advantageous to work in the complete coordinate representation, without any transformations to the momentum space. This is another difference with respect to the approach of Ohno and Okabe [36] who employ the full momentum-space representation in all d directions as it was originally done in the earliest field-theoretical work on semi-infinite systems [44,47].

The fundamental two-point correlation function of the free theory corresponding to Eq. (2.3) is given by the Dirichlet propagator

$$\langle \varphi_i(r,z) \varphi_j(0,z') \rangle_0 = G_D(r;z,z') \delta_{ij}. \quad (3.1)$$

In the familiar pz representation the Dirichlet propagator reads

$$G_D(p;z,z') = \frac{1}{2\kappa_0} [e^{-\kappa_0|z-z'|} - e^{-\kappa_0(z+z')}], \quad (3.2)$$

where the standard notation is used, $\kappa_0 = \sqrt{p^2 + m_0^2}$.

The Dirichlet propagator vanishes identically when at least one of its z coordinates is zero. Consequently, all the correlation functions involving at least one field at the surface vanish. This property holds for both the free and renormalized theories [30]. Owing to this property, it is not a straightforward matter to analyze the surface critical behavior at the ordinary transition.

In fact, the critical surface singularities at the ordinary transition can be extracted by studying the correlation functions with insertions of (inner) normal derivatives of the fields at the boundary, $\partial_n \varphi(r)$ [48,44,49]. Actually, in order to obtain the characteristic exponent $\eta_\parallel^{\text{ord}}$ of surface correlations, it is sufficient to consider a correlation function of two normal derivatives of boundary fields

$$\mathcal{G}_2(p) = \left\langle \frac{\partial}{\partial z} \varphi(p,z)\big|_{z=0} \frac{\partial}{\partial z'} \varphi(-p,z')\big|_{z'=0} \right\rangle, \quad (3.3)$$

where the fields $\varphi(p,z)$ are the Fourier transforms of the fields $\varphi(r,z)$ in $(d-1)$-dimensional parallel subspace. $\mathcal{G}_2(p)$ is a parallel Fourier transform of the corresponding two-point function $\mathcal{G}_2(r)$ in direct space. At the critical point $\mathcal{G}_2(p)$ behaves as $p^{-1+\eta_{\parallel}^{\mathrm{ord}}}$. It reproduces the leading critical behavior of a two-point function $G_2(p) = \langle \varphi(p,z)\varphi(-p,z')\rangle$ in the vicinity of the boundary plane. The surface critical exponent $\eta_{\parallel}^{\mathrm{ord}}$ is given by the scaling dimension of the boundary operator $\partial_n\varphi(r)$.

In the presence of randomness, the exponent $\eta_{\parallel}^{\mathrm{ord}}$ differs from its counterpart in ordered semi-infinite systems. The other surface exponents of the ordinary transition can be determined through the scaling laws [30].

In the present formulation of the problem, the renormalization process for the random system is essentially the same as in the "pure" case [30,35]. One introduces the renormalized bulk field and its normal derivative at the surface through

$$\varphi_R(x) = Z_\varphi^{-1/2}\varphi(x) \quad \text{and} \quad [\partial_n\varphi(r)]_R = Z_{\partial\varphi}^{1/2}\partial_n\varphi(r),$$
(3.4)

and renormalized correlation functions involving N bulk fields and M normal derivatives

$$\mathcal{G}_R^{(N,M)}(p;m,u,v) = Z_\varphi^{-N/2}Z_{\partial\varphi}^{-M/2}\mathcal{G}^{(N,M)}(p;m_0,u_0,v_0)$$
(3.5)

for $(N,M)\neq(0,2)$. In the special case of only two surface operators $(N,M)=(0,2)$, an additional, *additive* renormalization is required, so that

$$\mathcal{G}_R^{(0,2)}(p) = Z_{\partial\varphi}^{-1}[\mathcal{G}^{(0,2)}(p) - \mathcal{G}^{(0,2)}(p=0)].$$
(3.6)

Everywhere the renormalizations of the mass and coupling constants are implicit. These are the standard ones of the massive infinite-volume theory. The surface renormalization factor $Z_{\partial\varphi}(u,v)$ can be conveniently obtained from the consideration of the boundary two-point function $\mathcal{G}^{(0,2)}$, which is the object with the simplest Feynman graph expansion (see below):

$$Z_{\partial\varphi} = -\lim_{p\to 0}\frac{m}{p}\frac{\partial}{\partial p}\mathcal{G}^{(0,2)}(p).$$
(3.7)

A standard RG argument involving an inhomogeneous Callan-Symanzik equation yields the anomalous dimension of the operator $\partial_n\varphi(r)$

$$\eta_{\partial\varphi} = m\frac{\partial}{\partial m}\ln Z_{\partial\varphi}\big|_{\mathrm{FP}}$$

$$= \beta_u(u,v)\frac{\partial\ln Z_{\partial\varphi}(u,v)}{\partial u} + \beta_v(u,v)\frac{\partial\ln Z_{\partial\varphi}(u,v)}{\partial v}\bigg|_{\mathrm{FP}}.$$
(3.8)

"FP" indicates here that the above value should be calculated at the infrared-stable random fixed point of the underlying bulk theory $(u,v)=(u^*,v^*)$. The surface critical ex-

ponent $\eta_{\parallel}$ at the ordinary transition (here and further below we suppress the superscript "ord" at the surface critical exponents) is then given by

$$\eta_{\parallel} = 2 + \eta_{\partial\varphi}.$$
(3.9)

IV. TWO-LOOP APPROXIMATION

The Feynman diagram expansion of the unrenormalized correlation function $\mathcal{G}^{(0,2)}$ is, to the two-loop order,

(4.1)

Full internal lines represent here the free Dirichlet propagators (3.2) and dashed external lines give the factors $e^{-\kappa_0 z_i}$ when attached to the internal point with the coordinate z_i. Let us enumerate diagrams in the above sequence 1, 2, 3, 4. In the present theory described by the effective Hamiltonian (2.3), these graphs have their corresponding weights (aside from the standard symmetry factors)

$$-\frac{t_1^{(0)}}{2} \quad \text{with} \quad t_1^{(0)} = \frac{n+2}{3}u_0 + v_0,$$
(4.2a)

$$\frac{t_2^{(0)}}{6} \quad \text{with} \quad t_2^{(0)} = \frac{n+2}{3}u_0^2 + v_0^2 + 2v_0u_0,$$
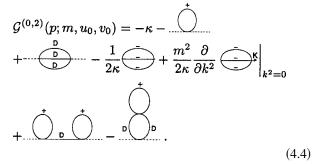(4.2b)

$$\frac{t_3^{(0)}}{4} \quad \text{and} \quad \frac{t_4^{(0)}}{4} \quad \text{with} \quad t_3^{(0)} = t_4^{(0)} = (t_1^{(0)})^2.$$
(4.2c)

The typical bulk short-distance singularities, which are present in the graphic expansion (4.1), are subtracted after performing the mass renormalization

(4.3)

Here the full lines with signs "$-$" denote the free bulk propagators. They are associated with the first term in the Dirichlet propagator (3.2), which is the usual bulk massive propagator in the pz representation. As a result of the mass renormalization one obtains

(4.4)

In this expansion the ''surface'' divergences still remain present, due to the one-loop self-energy insertions. These are represented by the closed lines with the index ''+'' which means including only the second, mirror term of the free propagator (3.2). Moreover, we have explicitly indexed here, with the label ''D,'' the original Dirichlet lines [see Eq. (4.1)]. Bulk propagators are denoted with ''−'' signs. The last two graphs in the second line represent just the usual bulk subtractions.

The boundary singularity in the one-loop diagram of the two-point surface correlation function has to be removed through the additive renormalization [zero-momentum subtraction (3.6)]. This, however, does not influence the calculation of the renormalization factor $Z_{\partial\varphi}$ which involves a momentum derivation [see Eq. (3.7)]. Surface divergences, present in each of the two last bubble graphs, mutually cancel. Actually, the whole combination in the third line of Eq. (4.4) vanishes identically as in Ref. [35]. Hence, applying the rule (3.7), we obtain

$$Z_{\partial\varphi} = 1 + \frac{\partial}{\partial\kappa}\ \overset{+}{\bigcirc}\ \Bigg|_{\kappa=m} - \lim_{p\to 0}\frac{m}{p}\frac{\partial}{\partial p}\ \overset{D}{\underset{D}{\ominus}}\ $$
$$-\frac{1}{2m^2}\left[\ \ominus - m^2\frac{\partial}{\partial k^2}\ \overset{\kappa}{\ominus}\ \Bigg|_{k^2=0}\right].$$
(4.5)

Performing the Feynman integrals in (4.5) by analogy with Ref. [35] we derive the result

$$Z_{\partial\varphi}(\bar{u}_0,\bar{v}_0) = 1 + \frac{\bar{t}_1^{(0)}}{4} + \bar{t}_2^{(0)}C,$$
(4.6)

where the constant C stems from the two-loop contribution into Eq. (4.5),

$$C \simeq \frac{107}{162} - \frac{7}{3}\ln\frac{4}{3} - 0.094299 \simeq -0.105063.$$
(4.7)

Here the renormalization factor $Z_{\partial\varphi}$ is expressed as a second-order series expansion in powers of bare dimensionless parameters $\bar{u}_0 = u_0/(8\pi m)$ and $\bar{v}_0 = v_0/(8\pi m)$. The corresponding weighting factors $\bar{t}_1^{(0)}$ and $\bar{t}_2^{(0)}$ are obtained by replacements $(u_0,v_0)\to(\bar{u}_0,\bar{v}_0)$ in the original combinations $t_1^{(0)}$ and $t_2^{(0)}$ from Eq. (4.2).

As it is usual in superrenormalizable theories, the renormalization factor expressed in terms of unrenormalized coupling constants is finite. As a next step, the vertex renormalizations should be carried out. To the present accuracy, they are given by

$$\bar{u}_0 = \bar{u}\left(1 + \frac{n+8}{6}\bar{u} + \bar{v}\right),$$
(4.8a)

$$\bar{v}_0 = \bar{v}\left(1 + \frac{3}{2}\bar{v} + 2\bar{u}\right).$$
(4.8b)

Again, the vertex renormalization at $d=3$ is a finite reparametrization. All relevant singularities have been removed already after the mass renormalization and after taking into account the special bubble-graph combinations emerging in presence of *Dirichlet* propagators. The result is a modified series expansion

$$Z_{\partial\varphi}(\bar{u},\bar{v}) = 1 + \frac{n+2}{12}\bar{u} + \frac{\bar{v}}{4} + \frac{n+2}{3}\left(C + \frac{n+8}{24}\right)\bar{u}^2$$
$$+ \left(C + \frac{3}{8}\right)\bar{v}^2 + 2\left(C + \frac{n+8}{24}\right)\bar{u}\bar{v}.$$
(4.9)

Combining the renormalization factor $Z_{\partial\varphi}(\bar{u},\bar{v})$ together with the one-loop pieces of the beta functions

$$\beta_{\bar{u}}(\bar{u},\bar{v}) = -\bar{u}\left(1 - \frac{n+8}{6}\bar{u} - \bar{v}\right),$$
(4.10a)

$$\beta_{\bar{v}}(\bar{u},\bar{v}) = -\bar{v}\left(1 - \frac{3}{2}\bar{v} - 2\bar{u}\right),$$
(4.10b)

through Eq. (3.8) yields the desired series expansion for $\eta_{\partial\varphi}$.

In terms of renormalized coupling constants u and v, normalized in a standard fashion so that

$$u = \frac{(n+8)}{6}\bar{u},\quad \frac{3}{2}\bar{v},$$

we obtain finally

$$\eta_{\partial\varphi}(u,v) = -\frac{n+2}{2(n+8)}u - \frac{v}{6} - 24\frac{(n+2)}{(n+8)^2}\mathcal{C}(n)u^2$$
$$-\frac{8}{9}\mathcal{C}(1)v^2 - \frac{16}{n+8}\mathcal{C}(n)uv,$$
(4.11)

where $\mathcal{C}(n)$ is a function of the replica number n, defined as

$$\mathcal{C}(n) = C + \frac{n+14}{96}.$$
(4.12)

In fact, the last expression (4.11) for $\eta_{\partial\varphi}$ provides a result for the *cubic anisotropic* model given by the effective Hamiltonian (2.3) with general number n of order-parameter components. In the case of infinite space, this last model attracted much attention very recently (see, e.g., Refs. [22], [50], [51], and references therein).

In the following we restrict our discussion to the case of random Ising system by taking the replica limit $n\to 0$. Hence, we obtain the next two-loop expansion for the surface critical exponent $\eta_\parallel(u,v)$

$$\eta_\parallel = 2 - \frac{u}{8} - \frac{v}{6} - \frac{3}{4}\mathcal{C}(0)u^2 - \frac{8}{9}\mathcal{C}(1)v^2 - 2\mathcal{C}(0)uv.$$
(4.13)

The numerical value of this exponent should be evaluated at the RG random fixed point of the underlying bulk theory [16]

$$u^* = -0.60509,\quad v^* = 2.39631.$$
(4.14)

Through the scaling relations we get access to the other surface critical exponents.

TABLE I. Surface critical exponents of the ordinary transition for $d=3$ up to two-loop order at the random-fixed point $u^*=-0.60509$, $v^*=2.39631$.

exp	$\dfrac{O_1}{O_2}$	[0/0]	[1/0]	[0/1]	[2/0]	[0/2]	[11/1]	[1/11]
$\eta_\parallel$	2.1	2.0	1.676	1.721	1.522	1.581	1.364	1.358
Δ_1	2.4	0.25	0.412	0.443	0.481	0.507	0.530	0.531
$\eta_\perp$	2.6	1.00	0.838	0.861	0.776	0.800	0.736	0.735
β_1	−3.1	0.75	0.912	0.943	0.860	0.841	0.875	0.876
γ_{11}	0.0	−0.50	−0.500	−0.500	−0.380	−0.364	–	–
γ_1	3.7	0.50	0.743	0.821	0.808	0.832	0.829	0.832
δ_1	1.9	1.67	1.847	1.868	1.941	1.968	2.056	2.054
δ_{11}	1.7	0.33	0.477	0.501	0.561	0.595	0.695	0.691

V. SURFACE CRITICAL EXPONENTS

As it is well known, the knowledge of one particular exponent at the ordinary transition implies the knowledge of the complete set of surface critical exponents through the scaling relations [30]. For convenience we quote them here:

$$\eta_\perp = \frac{\eta + \eta_\parallel}{2},$$

$$\beta_1 = \frac{\nu}{2}(d-2+\eta_\parallel),$$

$$\gamma_{11} = \nu(1-\eta_\parallel),$$

$$\gamma_1 = \nu(2-\eta_\perp),$$

$$\Delta_1 = \frac{\nu}{2}(d-\eta_\parallel),$$

$$\delta_1 = \frac{\Delta}{\beta_1} = \frac{d+2-\eta}{d-2+\eta_\parallel},$$

$$\delta_{11} = \frac{\Delta_1}{\beta_1} = \frac{d-\eta_\parallel}{d-2+\eta_\parallel}.$$

The exponent $\eta_\perp$ characterizes the critical-point correlations perpendicular to the surface, β_1 describes the decay of the surface magnetization on approaching the critical temperature, γ_{11} is the (local) surface susceptibility exponent, γ_1 is the layer susceptibility exponent, Δ_1 is the surface magnetic shift exponent, and δ_{11} and δ_1 give relations between the surface magnetization and the surface and bulk external magnetic fields, respectively, along the critical isotherm. The values ν, η, and $\Delta = \nu(d+2-\eta)/2$ are the standard bulk exponents.

In order to obtain individual RG expansions for each surface exponent, we use the above scaling laws (with $d=3$) combining Eq. (4.13) with the $n\rightarrow0$ limits of the two-loop series expansions for bulk exponents ν and η [17,16,52]

$$\nu = \frac{1}{2} + \frac{n+2}{4(n+8)}u + \frac{v}{12} - \frac{1}{108}\left[\frac{(n+2)(38-27n)}{2(n+8)^2}u^2\right.$$
$$\left. + \frac{11}{54}v^2 + \frac{(38-27n)}{3(n+8)}uv\right], \tag{5.1}$$

$$\eta = \frac{8}{27}\left[\frac{n+2}{(n+8)^2}u^2 + \frac{v^2}{27} + \frac{2uv}{3(n+8)}\right]. \tag{5.2}$$

For each of surface critical exponents we obtain a double series expansion in powers of u and v, truncated at the second order. Power series expansions of this kind are known to be generally divergent due to a nearly factorial growth of expansion coefficients at large orders of perturbation theory [53–56]. Hence, the numerical evaluation of the exponents represented by such series expansions requires "resummation" procedure.

The simplest way to obtain meaningful and rather accurate numerical estimates is to construct a table of rational approximants in two variables from the original series expansions. This should work already well when the series behave in lowest orders "in a convergent fashion." Apparently divergent ones require more sophisticated summation procedures.

The results of our Padé analysis are represented in Table I. To give an idea about relative magnitudes of first-order (O_1) and second-order (O_2) perturbative corrections appearing in our series expansions, we quote their ratio O_1/O_2 (at fixed point) in the second column. The larger (absolute) values of this ratio correspond to the better apparent convergence of truncated series. Except for β_1 and γ_{11}, the values of O_1/O_2 are positive. This means that the signs of the first- and second-order corrections do not alternate for most of the exponents. Note, that a very similar situation has been encountered in the analysis of perturbation expansions for surface exponents at the ordinary transition in pure systems [35].

In fact, as it has been shown in Ref. [35], the best numerical estimates for surface exponents in pure semi-infinite sys-

tems have been given by diagonal [1/1] Padé approximants. In the present case the qualitative behavior of underlying series expansions is very similar. In accordance with this we expect to obtain numerical results of comparable rather good quality from nearly diagonal two-variable rational approximants of the types

$$[11/1] = \frac{1 + a_1 u + \bar{a}_1 v + a_{11} uv}{1 + b_1 u + \bar{b}_1 v} \tag{5.3}$$

and

$$[1/11] = \frac{1 + a_1 u + \bar{a}_1 v}{1 + b_1 u + \bar{b}_1 v + b_{11} uv}. \tag{5.4}$$

The corresponding values of surface critical exponents are given in the last two columns of Table I. As we can see, these numbers do not differ significantly between themselves.

The values [0/0], [1/0], and [2/0] are simply the direct partial sums up to the zeroth, first, and second orders, respectively. We consider the [11/1] and [1/11] values as the best we could achieve from the available knowledge about the series expansions in the frames of the present approximation scheme. Their deviations from the other second-order estimates of the table might serve as a rough measure of the achieved numerical accuracy.

VI. $\sqrt{\varepsilon}$ EXPANSION

An alternative approach to calculate the desired surface critical exponents is the treatment of the theory in $4-\varepsilon$ space dimensions ($\sqrt{\varepsilon}$ expansion) and subsequent extrapolation to $\varepsilon = 1$. This approach was initiated by Ohno and Okabe [36]. These authors considered the two-loop approximation for correlation functions in random semi-infinite space. They derived the corresponding series expansions in powers of renormalized coupling constants and ε, for surface exponents $\eta_{\parallel}$ and $\eta_{\perp}$. We quote their results for the ordinary transition in the case of present interest, setting $n = 1$ in the expressions of Ref. [36] and changing normalizations of coupling constants ($u \rightarrow v/24$, $w \rightarrow -u/3$) to fit with our notations:

$$\eta_{\parallel} = 2 - \frac{u}{3} - \frac{v}{2} + \frac{u^2}{4} + \frac{5}{12} v^2 + \frac{3}{4} uv + \cdots, \tag{6.1}$$

$$\eta_{\perp} = 1 - \frac{u}{6} - \frac{v}{4} + \frac{5}{36} u^2 + \frac{11}{48} v^2 + \frac{5}{12} uv + \cdots. \tag{6.2}$$

In the present problem dots represent less important terms of order $O(\varepsilon^{3/2})$. Unfortunately, only the first nontrivial corrections $\propto \sqrt{\varepsilon}$ have been obtained from these equations in Ref. [36].

Actually, it is possible to derive one more term in the $\sqrt{\varepsilon}$ expansion of surface critical exponents using the known expressions for the fixed-point values up to $O(\varepsilon)$ [13,14]

TABLE II. Surface critical exponents of the ordinary transition from the $\sqrt{\varepsilon}$ expansion.

exp	$\dfrac{O_1}{O_2}$	[0/0]	[1/0]	[0/1]	[2/0]	[0/2]	[1/1]
$\eta_{\parallel}$	-2.1	2.00	1.664	1.712	1.824	1.792	1.772
$\eta_{\perp}$	-2.2	1.00	0.832	0.856	0.907	0.892	0.884
β_1	37.6	0.75	0.792	0.794	0.793	0.793	0.793
γ_{11}	-2.4	-0.50	-0.416	-0.408	-0.451	-0.457	-0.441
γ_1	-4.2	0.50	0.668	0.702	0.628	0.611	0.636

$$u^* = -3 \sqrt{\frac{6\varepsilon}{53}} + 18 \frac{110 + 63\zeta(3)}{53^2} \varepsilon, \tag{6.3}$$

$$v^* = 4 \sqrt{\frac{6\varepsilon}{53}} - 72 \frac{19 - 21\zeta(3)}{53^2} \varepsilon, \tag{6.4}$$

where $\zeta(3) \simeq 1.2020569$ is the Riemann ζ function, and the usual geometric factor $K_d = 2^{1-d} \pi^{-d/2}/\Gamma(d/2)$ is absorbed into coupling constants. Thus we obtain

$$\eta_{\parallel} = 2 - \sqrt{\frac{6\varepsilon}{53}} + \frac{756\zeta(3) - 5}{2 \times 53^2} \varepsilon, \tag{6.5}$$

$$\eta_{\perp} = 1 - \frac{1}{2} \sqrt{\frac{6\varepsilon}{53}} + \frac{378\zeta(3) - 29}{2 \times 53^2} \varepsilon. \tag{6.6}$$

It can be easily verified that the above exponents satisfy the scaling relation $\eta_{\perp} = (\eta + \eta_{\parallel})/2$ with the correct value

$$\eta = -\frac{\varepsilon}{106} + O(\varepsilon^{3/2}) \tag{6.7}$$

of the bulk theory.

Taking into account scaling relations for surface critical exponents and $\sqrt{\varepsilon}$ expansions for random bulk exponents ν and η [13,14] we obtain, in addition,

$$\gamma_{11} = -\frac{1}{2} + \frac{1}{4} \sqrt{\frac{6\varepsilon}{53}} - \frac{3}{8} \frac{[252\zeta(3) - 37]}{53^2} \varepsilon, \tag{6.8}$$

$$\gamma_1 = \frac{1}{2} + \frac{1}{2} \sqrt{\frac{6\varepsilon}{53}} + \frac{6211 - 1512\zeta(3)}{8 \times 53^2} \varepsilon, \tag{6.9}$$

$$\beta_1 = \frac{3}{4} + \frac{1}{8} \sqrt{\frac{6\varepsilon}{53}} - \frac{7}{16} \frac{[108\zeta(3) - 137]}{53^2} \varepsilon, \tag{6.10}$$

Similarly as in the case $d = 3$ of the previous section, we performed a Padé analysis of our $\sqrt{\varepsilon}$ expansions at $\varepsilon = 1$. The numerical values of surface critical exponents obtained in this way are represented in Table II.

The [1/0] values for the exponents $\eta_{\parallel}$ and $\eta_{\perp}$ reproduce the first-order results by Ohno and Okabe [36]. On the other hand, the other exponents, β_1, γ_{11}, and γ_1 slightly differ. The reason is that we calculated our [1/0] estimates directly

from each respective $\sqrt{\varepsilon}$ expansion, while in Ref. [36] they were obtained from the scaling relations using the above numerical values of $\eta_\parallel$ and $\eta_\perp$.

Comparing the results from Tables I and II we see that the values of first-order approximants, denoted by [1/0] and [0/1] in both cases, are of comparable magnitudes. But, on the other hand, the values from second-order approximants are significantly different in both tables. The reason is that the second-order contributions of the $\sqrt{\varepsilon}$ expansion provide corrections of opposite signs, as compared to the ''three-dimensional'' theory of the previous section (corresponding orders' ratios O_1/O_2 have opposite signs).

Our choice will be in favor of our estimates derived directly in three dimensions for the following reasons. It is well known that to order ε, the $\sqrt{\varepsilon}$ expansion fails to yield the positive correlation function exponent η for the random bulk Ising system [see Eq. (6.7)]. In fact, if we try, using the scaling relation $\eta = 2\eta_\perp - \eta_\parallel$, to reproduce the numerical value of η from our second-order data of Table II, we also always obtain negative values. This deficiency is not present in our calculations directly at $d=3$. Moreover, there are several reports in the literature on ''bad'' behavior of the $\sqrt{\varepsilon}$ expansion at larger orders, and for other bulk exponents [20,21,23].

At the same time, the fixed-dimension massive field theory appears to give quite regular and reliable results for random bulk systems in three dimensions [21–24], even at rather low orders of perturbation theory [16,57,18,58]. From our experience, the massive field theory works also well in description of surface critical behavior in pure three-dimensional systems [42,35].

VII. SUMMARY

We have studied the surface critical behavior of three-dimensional quenched random semi-infinite Ising systems with free plane boundaries. We have calculated the corresponding surface critical exponents of the ordinary transition employing two alternative possibilities: (a) the massive field-theoretic approach directly in $d=3$ dimensions and (b) the $\sqrt{\varepsilon}$ expansion about the upper critical dimension $d=4$ with subsequent extrapolation to $\varepsilon=1$. In this latter calculation we extend, to the order of $O[(\sqrt{\varepsilon})^2]$, the previous first-order results by Ohno and Okabe [36].

We performed a rational approximant (Padé) analysis of the resulting perturbation series expansions in both cases attempting to find out the best numerical values of surface exponents in three dimensions. However, the typical behavior of perturbative expansions in both calculational schemes appeared to be qualitatively different.

In the last section we gave some arguments in favor of results obtained in the framework (a): directly at $d=3$. Thus, the summary for our final numerical values of the surface critical exponents at the ordinary transition in the presence of randomness is

$$\eta_\parallel = 1.36, \quad \Delta_1 = 0.53, \quad \eta_\perp = 0.74, \quad \beta_1 = 0.88,$$

$$\gamma_{11} = -0.24, \quad \gamma_1 = 0.83, \quad \delta_1 = 2.1, \quad \delta_{11} = 0.69.$$

The above values stem from the last two columns of Table I. The estimate of γ_{11}, for which no approximants [11/1] and [1/11] exist, has been derived from the scaling relation $\gamma_{11} = \nu(1 - \eta_\parallel)$, where we used $\nu = 0.68$ [16] and the above value of $\eta_\parallel = 1.36$. We believe, these values are the best one can derive from all above calculations.

The values of our exponents characterizing the surface critical behavior of semi-infinite quenched random Ising-like systems are apparently different from their counterparts of pure Ising systems [42,35]: $\eta_\parallel = 1.528$, $\Delta_1 = 0.464$, $\eta_\perp = 0.779$, $\beta_1 = 0.796$, $\gamma_{11} = -0.333$, $\gamma_1 = 0.769$, $\delta_1 = 1.966$, $\delta_{11} = 0.582$.

We quantitatively confirm a general expectation that the change of the bulk universality class of a system should affect its boundary critical behavior. So, the semi-infinite systems with quenched bulk disorder are characterized by a new set of the surface critical exponents. We suggest that the obtained results could stimulate further experimental work as well as numerical investigations of the boundary critical behavior of disordered systems.

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